Magnetohydrodynamic flows of a perfectly conducting, viscous fluid

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The paper considers the flow of an incompressible, viscous, perfectly conducting fluid past a fixed obstacle in the presence of an applied magnetic field which is parallel to the stream at large distances from the obstacle. A simple transformation of the fluid velocity and the total head enables the magnetohydrodynamic flow past the obstacle to be determined from the corresponding flow of a non-conducting fluid past the same obstacle but with a reduced main-stream velocity. The method is illustrated by considering the flows past a sphere, a circular cylinder and a semi-infinite flat plate for different field strengths. The drag on the sphere is plotted as a function of the field strength for a fixed Reynolds number. The patterns of the flow past a circular cylinder are sketched and an inference is made to the way in which disturbances can propagate upstream for the case when the main-stream velocity is less than the Alfvén speed. These give rise in the first instance to a separation bubble upstream of the cylinder. Finally the range of applicability of familiar high Reynolds number approximations to magnetohydrodynamic flows is discussed. In particular, if the main-stream velocity is equal to the Alfvén speed, the boundary-layer approximation is shown to be no longer valid.

1. Introduction

This work was initiated by some rather surprising results obtained by Greenspan & Carrier (1959) in their considerations of the magnetohydrodynamic flow past a semi-infinite flat plate. They examined the flow when the applied magnetic field and free-stream directions are aligned and uniform at large distances. The description of the flow depended on two parameters, namely the ratio β of the square of the Alfvén speed to the square of the main-stream velocity, and the product of the electrical conductivity σ and the kinematic viscosity ν. They found that when $\beta = 1$ the drag on any finite length of the plate became zero. This was interpreted as meaning that the magnetic field had become so distorted as to produce a magnetic wall which 'plugged' the entire flow. Part of the present discussion endeavours to question this result for the case when the fluid is perfectly conducting. It must, however, be pointed out that no alternative solution is proposed here for reasons which will become obvious later.

The present discussion covers a more general field of application to fluid flows than the problem referred to above. Here we shall consider steady magnetohydrodynamic flows past any fixed axisymmetric or two-dimensional obstacle.

We shall assume the fluid to be perfectly electrically conducting and consider only those problems for which the applied magnetic field and the undisturbed stream velocity are parallel at infinity. A transformation of the fluid velocity and total head in the equations of motion enables a magnetohydrodynamic flow of the above type to be obtained from the corresponding flow of a viscous *non-conducting* fluid, for which the well-known Navier–Stokes equations of motion are applicable. An interesting feature of the transformation is the relation existing between the Reynolds numbers of the two problems. This shows that as $\beta \to 1$, for a fixed value of the Reynolds number R in the magnetohydrodynamic flow problem, the Reynolds number R^N of the corresponding non-conducting fluid flow tends to zero. Indeed the equations of motion describing the magnetohydrodynamic flow then reduce almost to the Stokes equations of motion for slow flow.

The method is applied to the magnetohydrodynamic flows past a sphere, a circular cylinder and a semi-infinite flat plate. In the first of these, a plot of the drag of the sphere on the flow for varying magnetic field strength and a fixed Reynolds number is given. The graph indicates that the drag reaches a minimum value at $\beta = 1$. The patterns of the flow past a circular cylinder for varying values of β are also sketched. For $\beta > 1$, which corresponds to the undisturbed fluid stream moving with a velocity less than the Alfvén speed, the transformation indicates that the flow in the corresponding non-conducting fluid problem is reversed. Thus, for moderate values of R^N, the separation bubble, which in conventional fluid flows occurs downstream of the cylinder, now appears upstream in the magnetohydrodynamic flow and corresponds to the upstream influence of Alfvén waves together with diffusion and convection. It is well known that, for purely viscous fluid flows about a sphere and a circular cylinder, the flows become unstable at certain critical Reynolds numbers. The transformation would seem to indicate that the effect of the magnetic field is to stabilize the flow and increase the values of the critical Reynolds numbers in the magnetohydrodynamic flows. It should however be pointed out that a fuller discussion of the unsteady flow, to which the simple transformation cannot be applied, is needed on this point before definite values can be given.

Finally, a discussion of the flow past a semi-infinite flat plate is presented, though as a comment on already existing analyses it lacks authority because so little is known concerning the purely viscous fluid flow near the leading edge of the plate (i.e. flow at small Reynolds number).

2. The transformation

The equations governing the steady motion of an incompressible, viscous, electrically conducting fluid can be written in the form

$$\operatorname{div} \mathbf{q} = 0, \tag{1}$$

$$\operatorname{curl} \mathbf{q} \wedge \mathbf{q} = -\operatorname{grad}(p/\rho + \tfrac{1}{2}q^2) + \nu\nabla^2\mathbf{q} + (\mu/\rho)\,\mathbf{j} \wedge \mathbf{H}, \tag{2}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mu\mathbf{q} \wedge \mathbf{H}), \quad \operatorname{curl} \mathbf{H} = 4\pi\mathbf{j}, \tag{3, 4}$$

$$\operatorname{div} \mathbf{H} = 0, \quad \operatorname{curl} \mathbf{E} = 0. \tag{5, 6}$$

In the above equations $\mathbf{q}$, p and ρ denote the fluid velocity, pressure and density, respectively, $\mathbf{j}$ denotes the current density, $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields.

The present discussion is limited to magnetohydrodynamic flow problems in which the magnetic field is applied parallel to the direction of the free stream at infinity. A further restriction becomes necessary later, in choosing the ratio of the magnitudes of the applied magnetic field and the free-stream velocity to be uniform. Equation (3) shows that when the ambient magnetic field and fluid stream are aligned, the electric field is zero at infinity. Assuming the flow to be axisymmetric or two-dimensional, the electric field satisfies the equation

$$\operatorname{div} \mathbf{E} = 0. \tag{7}$$

This equation, together with equation (6) and the boundary conditions, is sufficient to show that, in the magnetohydrodynamic flow past an uncharged obstacle of the type described above, the electric field will be zero everywhere.

For a perfectly conducting fluid, in which the electrical conductivity σ is infinite, equation (3) with $\mathbf{E} \equiv 0$ shows that either $\mathbf{q}$ or $\mathbf{H}$ must be zero, or $\mathbf{q}$ and $\mathbf{H}$ must be parallel everywhere. The former result is trivial, the latter leads to the relation

$$\mathbf{H} = f(x, y, z)\, \mathbf{q}, \tag{8}$$

where $f(x, y, z)$ is a function which is constant along a streamline (or magnetic-field line), since, from equations (5) and (1),

$$0 = \operatorname{div} \mathbf{H} = \mathbf{q}\,.\operatorname{grad} f + f \operatorname{div} \mathbf{q} = \mathbf{q}\,.\operatorname{grad} f. \tag{9}$$

If the ratio of the magnitudes of the magnetic field H_0 and the fluid velocity U_0 at infinity is uniform, then f is equal to H_0/U_0 along all streamlines originating at infinity. Substituting for $\mathbf{H}$ in equations (2) and (4), we obtain

$$(1-\beta) \operatorname{curl} \mathbf{q} \wedge \mathbf{q} = -(1/\rho) \operatorname{grad} h + \nu \nabla^2 \mathbf{q}, \tag{10}$$

where $h = p + \frac{1}{2}\rho q^2$ is the total head and $\beta = \mu H_0^2 / 4\pi\rho U_0^2$.

We now transform the velocity and total head according to the relations

$$\mathbf{q} = [1/(1-\beta)]\, \mathbf{q}^N, \quad h = [1/(1-\beta)]\, h^N + C, \tag{11}$$

where C is a constant. $\mathbf{q}^N$ and h^N then satisfy the Navier–Stokes equations of flow for a viscous incompressible *non-conducting* fluid, namely

$$\operatorname{div} \mathbf{q}^N = 0, \tag{12}$$

$$\operatorname{curl} \mathbf{q}^N \wedge \mathbf{q}^N = -(1/\rho) \operatorname{grad} h^N + \nu \nabla^2 \mathbf{q}^N. \tag{13}$$

Hence for problems in which the ambient magnetic field and the free-stream directions are aligned, the pattern of the flow of a perfectly conducting fluid past an obstable is the same as that of a non-conducting fluid flowing with velocity $(1-\beta)\, U_0$ past the same obstacle. The superscript N will be used to label quantities in the corresponding non-conducting fluid-flow problem. The total head at a point in the fluid is found from the corresponding non-conducting fluid-flow value by the use of transformation (11), the constant C being determined by the conditions at infinity. The fluid pressure is given by the relation

$$p-p_0 = (1-\beta)^{-1}(p^N-p_0^N)-\tfrac{1}{2}\rho\beta(1-\beta)^{-1}[(q^N)^2-(1-\beta)U_0^2], \tag{14}$$

where p_0 and p_0^N are the corresponding values of the pressures at infinity.

The reader will notice that the above results could have been established by a transformation of either the length scale or the viscosity instead of the velocity.

Several interesting features become apparent when the Reynolds number, $R = U_0 L/\nu$, in the magneto-hydrodynamic-flow problem is related to the Reynolds number, $R^N = U_0^N L/\nu$, in the analogous 'non-conducting' fluid-flow problem; they are connected by the equation

$$R^N = |1-\beta|\, R, \tag{15}$$

where the modulus signs have been inserted for obvious reasons. Thus, if R is fixed and β is increased from 0 to 1, the Reynolds number R^N in the corresponding non-conducting fluid-flow problem decreases from R to 0. One must therefore tread cautiously near $\beta = 1$, for this corresponds to non-conducting fluid flows in which the Reynolds number R^N is small. In particular it must be noted that in this region the boundary-layer approximation can no longer be used to determine the magnetohydrodynamic flow, since the validity of the approximation requires that $R^N = |1-\beta|R$ should be large and not just simply that R should be large. A fuller discussion of this point and other approximations will be given later. In the limit as $\beta \to 1$ from below, equation (10) shows that the Navier–Stokes equations of magnetohydrodynamic flow reduce to the classical Stokes equations, provided that they are written in a form involving the total pressure rather than the static pressure. These equations are used to treat problems involving slow flow in which the Reynolds number is low. This is consistent with the transformation (15) which shows that as $\beta \to 1$ the Reynolds number in the analogous non-conducting fluid flow problem becomes small. We must emphasize, however, that in the magnetohydrodynamic problem for which $\beta = 1$ Stokes's equations are the exact form of the full Navier-Stokes equations of magnetohydrodynamic flow provided that the total head is used instead of the static pressure. The value of $\beta = 1$ is in the nature of a critical point as it marks the transition from fluid flows having super-Alfvén speed to those having sub-Alfvén speed. For $\beta > 1$, the transformation shows that the flow in the 'non-conducting fluid' problem is reversed. This will be illustrated later.

As an example of the use of the transformation described in this section we now determine the magnetohydrodynamic flow past a sphere. Clearly a similar analysis could be performed for the circular cylinder and other shapes.

3. Magnetohydrodynamic flow past a sphere

Several attempts have been made to consider this problem for fluids having low electrical conductivity. In particular those of Chester (1957) and Blerkom (1960) are illuminating. We are, however, concerned with a perfectly conducting fluid. Blerkom did include some discussion of this case in his paper and he noticed the correspondence exhibited in equation (15) by examining the Oseen linearized equations extended to include the effects of a magnetic field. He did not, however, pursue the matter further.

A study of the magnetohydrodynamic flow past a sphere has several advantages; in particular the drag of the sphere in a non-conducting fluid flow is known experimentally as a function of the Reynolds number. In addition, the Stokes solution describing the slow flow past a sphere is known.

The use of relation (15) and the experimental information permit a very simple method of determining the drag of a sphere in a magnetohydrodynamic flow of the type already described. The experimental results, which have been used here,

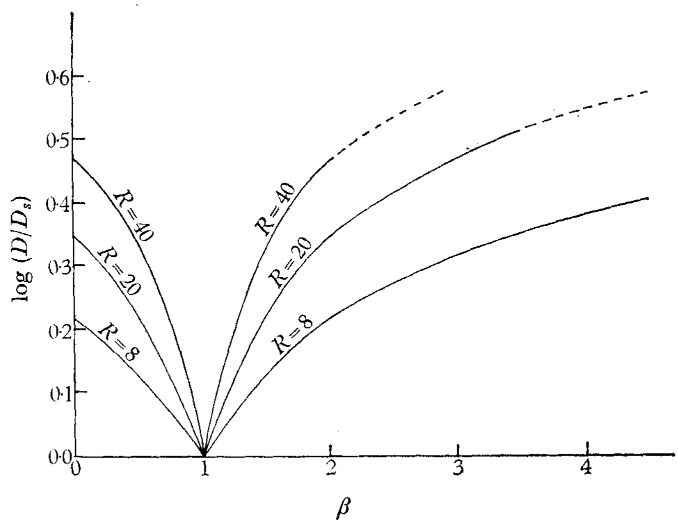

FIGURE 1. The ratio of the drag D to the Stokes value D_s plotted as a function of the Alfvén number β for values of the Reynolds number $R = 40$, 20 and 8.

are those given by Goldstein (1938), who plots the drag coefficient C_D^N in the non-conducting fluid flow problem as a function of $R^N = 2U^N a/\nu$, where a is the radius of the sphere. In order to find how the drag of a sphere in the magnetohydrodynamic flow depends upon the strength of the applied magnetic field, we fix the value of R and vary β, thus obtaining varying values of R^N from equation (15). The values of C_D^N are then determined from the graph of C_D^N against R^N. The drag D in the magnetohydrodynamic problem is then calculated from a knowledge of C_D^N in the following way:

$$D = (1-\beta)^{-1} D^N = \tfrac{1}{8}\rho\pi\nu^2(1-\beta)^{-1}(R^N)^2 C_D^N = \tfrac{1}{8}\rho\pi\nu^2(1-\beta) R^2 C_D^N. \qquad (16)$$

In figure 1 the total drag D is plotted as a function of β for values of R equal to 8, 20 and 40. For $\beta > 1$, the flow in the corresponding non-conducting fluid problem is reversed, so that the drag D^N will act in the opposite direction to that of the free stream in the magnetohydrodynamic problem. However, the drag D is still positive, since the negative factor $(1-\beta)$ in equation (16) means that D is in the direction opposite to D^N.

The variation of the drag in the neighbourhood of the critical point $\beta = 1$ can be discussed by using the theoretical results of Proudman & Pearson (1957), who considered the slow flow of a viscous fluid past a sphere. They gave the drag coefficient for small values of R^N as

$$C_D^N = \frac{24}{R^N}\{1+\tfrac{3}{16}R^N+\tfrac{9}{160}(R^N)^2\log R^N+O(R^N)^2\}. \tag{17}$$

Applying the transformation rules and substituting in equation (16), we obtain

$$D = 6\pi\rho\nu a U_0\{1+\tfrac{3}{16}|1-\beta|R+\tfrac{9}{160}(1-\beta)^2R^2\log|1-\beta|R+O[(1-\beta)^2R^2]\}. \tag{18}$$

The above expression will hold for both sufficiently small values of R and of $|1-\beta|$. For a fixed value of R as $\beta \to 1$ the drag reaches a minimum value, which is the classical Stokes value; this was anticipated in our earlier discussion. At the critical point, $\beta = 1$, the slope of the drag curve is discontinuous. This is due to the different character of the flow in changing from super-Alfvén to sub-Alfvén flow, in which Alfvén waves can propagate upstream. These could affect the flow at infinity. However, when $\beta > 1$, the corresponding flow in the non-conducting fluid problem is reversed. For low values of $(\beta-1)R$ the non-conducting fluid flow solution would suggest that the uniform conditions upstream at infinity are not affected. This will be illustrated in the next section. For higher values of $(\beta-1)R$ the flow might contain a narrow wake in the magnetohydrodynamic flow extending upstream from the body (the wake in the *reversed* non-conducting fluid flow problem is downstream), in which case the uniform conditions upstream at infinity would be upset. This situation has been discussed by Stewartson (1960). It must be pointed out though that, if in the magnetohydrodynamic flow problem the conditions downstream at infinity were uniform, the transformation could be applied provided that the flow was steady. There seems to be no reason to suppose that this is inferior to the more usual reversed situation. It must however be mentioned that for a fluid having finite electrical conductivity there would also be a wake downstream. This has not been considered in the present paper. One further remark should also be made concerning the stability of the flow. It is well known both from experiment and theory that, for viscous non-conducting fluid flows of the type described, there is a critical Reynolds number for which the flow becomes unsteady. Experimentally this occurs for the flow about a sphere at a Reynolds number of about 100. Thus, according to the transformation (15), the magnetohydrodynamic flow might become unsteady at a Reynolds number of about $100/|1-\beta|$ from which we might infer that, for $0 < \beta < 2$, the effect of the magnetic field is to stabilize the flow. However, since the transformation is limited to steady flows only, a fuller investigation should be carried out before reaching any definite conclusions on this point.

4. Variation of the flow pattern with β

In order to discuss flow patterns for varying magnetic field strengths it is convenient to consider the uniform flow past a circular cylinder, since numerical computations of the corresponding flows of a viscous non-conducting fluid have been made for values of $R^N = 10$, 20 and 40 by Thom (1933) and Kawaguti (1953).

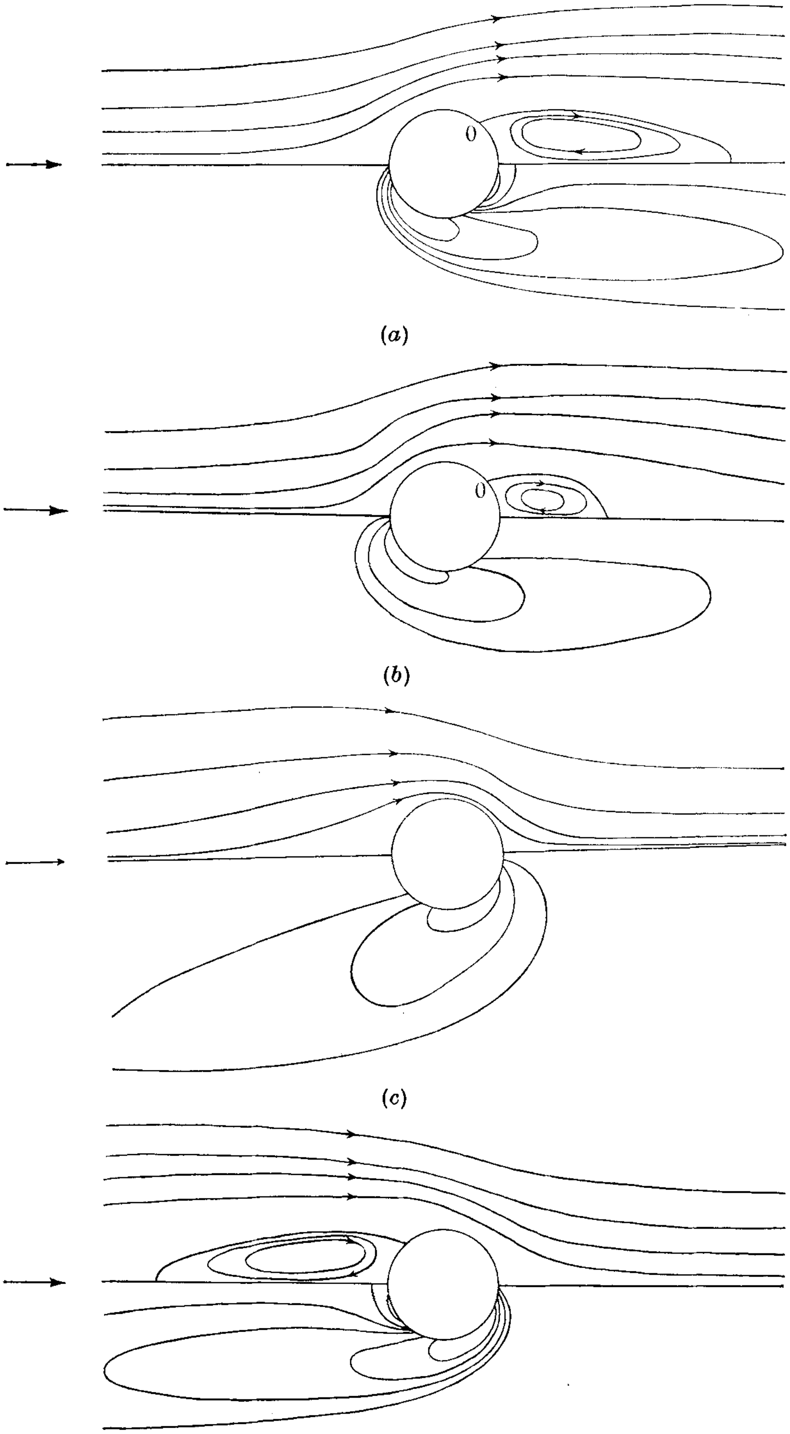

Figure 2. Streamlines (or magnetic lines) and lines of equal vorticity (or current) for the flow about a circular cylinder with $R = 40$ and $\beta = 0, \frac{1}{2}, \frac{5}{4}$ and 2. (*a*) $\beta = 0$, $R^N = 40$ Kawaguti 1953). (*b*) $\beta = \frac{1}{2}$, $R^N = 20$ (Thom 1933). (*c*) $\beta = \frac{5}{4}$, $R^N = 10$ (Thom 1933). (*d*) $\beta = 2$, $R^N = 40$ (Kawaguti 1953).

As long as $R^N = |1-\beta| R$ is less than about 40, the non-conducting fluid flow remains steady and we can apply the transformation procedure. Thus if we fix $R = 40$, the above computations will give immediately the magnetohydrodynamic flow patterns for values of $\beta = 0, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}$ and 2. In figure 2, the streamlines (or magnetic lines) and lines of equal vorticity (or current) are shown for $\beta = 0, \frac{1}{2}, \frac{5}{4}$ and 2.

As the magnetic field increases, the separation bubble containing the two standing eddies becomes smaller until it eventually disappears for β sufficiently near unity. As β increases from 1, the separation bubble appears upstream and grows with increasing β; ultimately the flow will become unsteady. There is no obvious reason for the separation bubble to behave in this way with increasing magnetic field strength. Clearly it is linked with the factors which govern the length of the separation bubble in the purely viscous fluid-flow problem. Unfortunately for the purpose of this paper no simple explanation has been offered. Presumably its length in the latter problem depends on both convection and diffusion, and the effect of the magnetic field in the magnetohydrodynamic flow is to decrease the convective part, as equation (10) shows, by the presence of the factor $1-\beta$. As the magnetic field increases, the Lorentz force becomes more dominant and produces a 'convective' rate in the opposite direction to the stream velocity. It is well known, in fact, that in a magnetic field the vorticity is not convected with the stream, but rather moves relative to the stream at the Alfvén speed along the magnetic lines of force. Thus in sub-Alfvén flow ($\beta > 1$), the wake must stretch upstream as well as downstream. It has already been mentioned that, if finite but large conductivity had been considered, then there would also be a wake on the downstream side as well as the upstream side. Blerkom (1960) has discussed this situation for the case of the flow past a sphere by applying the Oseen linearized treatment to the equations of motion.

5. Flow past a flat plate

Several attempts have been made to discuss the magnetohydrodynamic flow past a flat plate. In particular Greenspan & Carrier (1959) have discussed the flow when the magnetic-field and free-stream directions are aligned and uniform at large distance from the plate. They examined the flows for the full range of values of $\epsilon = \sigma\mu\nu$ and β and found that, when $\beta \geqslant 1$, no steady flow which is uniform at large distance from the plate is possible. Their calculations were based on two treatments, namely the use of the asymptotic form of the solution of the Navier–Stokes equations in terms of parabolic co-ordinates and the solution of the classical Oseen linearized equations. They then predicted that at $\beta = 1$ the drag on any finite length of the plate was zero. This was interpreted as meaning that the magnetic field had been so distorted as to produce a magnetic wall which 'plugged' the entire flow. By the use of the transformation described in the previous sections, we can discuss this situation for the case of a perfectly electrically conducting fluid. To find an approximation for the drag on the plate we use a result due to Imai (1957), who considered the viscous flow past a flat plate by using a series solution in inverse powers of the square root of the Reynolds number $R_x^N = U^N x/\nu$ based upon the distance x from the leading edge. The full solution also included logarithmic terms, but the retention of the first two terms in the expansion for the drag will suffice for the present discussion: this gives

$$D^N = \tfrac{1}{2}\rho x (U^N)^2 \left[\frac{1\cdot328}{(R_x^N)^{\frac{1}{2}}} + \frac{2\cdot326}{R_x^N} + \ldots\right] \tag{19}$$

for sufficiently large R_x^N.

Applying the transformation, the drag D of the flat plate in magnetohydrodynamic flow is obtained,

$$D = \frac{D^N}{1-\beta} = \tfrac{1}{2}\rho x U^2 \left[\frac{1\cdot328(1-\beta)^{\frac{1}{2}}}{R_x^{\frac{1}{2}}} + \frac{2\cdot326}{R_x} + \ldots\right]. \tag{20}$$

The first term only in expression (20) was obtained by Greenspan & Carrier, from which they concluded that as $\beta \to 1$ the drag became zero. It is seen, however that the above expression is no longer valid as $\beta \to 1$, since the above approximate treatment requires that $R_x^N = (1-\beta)R_x$ should be large. For the same reason, one also questions the applicability of the Oseen approximation to the problem in the region of the critical point which corresponds to slow flow in the analogous non-conducting fluid-flow problem. Since there does not exist an adequate theory which deals successfully with the slow flow past a semi-infinite flat plate, that is in the region of the leading edge, a fuller discussion is prevented at present. What can, however, be said with certainty is that any attempt to discuss the critical region about $\beta = 1$ by making the usual boundary-layer approximation will be doomed to failure.

For the case $\beta > 1$, it is immediately obvious why no solution could be found for the flow past a semi-infinite flat plate, since the transformation indicates that the flow in the corresponding non-conducting fluid problem is reversed, that is the flow will be travelling from right to left over each side of the semi-infinite flat plate ($0 \leqslant x \leqslant \infty$). Thus in the magnetohydrodynamic problem, no steady flow which is uniform at a large distance upstream of the plate could be possible in such a case. If one had considered a flat plate of finite length then a solution would exist and there would be a 'wake' upstream of the leading edge of the plate, the extent of the wake depending on the magnitude of $(\beta-1)R$, and very little disturbance downstream of the trailing edge. Greenspan & Carrier (1959) revealed this effect, and found that, for finite conductivity and $\beta > 1$, the effects of the plate were as prominent upstream as they were downstream. For infinite conductivity the effect is most prominent upstream. Greenspan (1960) has considered this effect further.

6. Conclusions

Though the theory advanced in this paper is simple and enables one to predict the possible behaviour of a perfectly electrically conducting fluid past a fixed obstacle, the author is aware of some of the difficulties which it presents. In particular, in obtaining the drag of a sphere and the patterns of the flow past a circular cylinder, it has been tacitly assumed that β remains constant *throughout* the entire flow. The reader is reminded that this depended on the choice of a constant value for the function f in equation (8). This can be justified for the case when all streamlines originate from a uniform region at infinity. However, in the flows which have been illustrated, a separation bubble has occurred and here the streamlines are closed. On these streamlines the function f, though constant *along* them, cannot be determined explicitly on the simple basis of steady flow and in assuming the electrical conductivity to be large. This indeterminancy is similar to the one experienced in the steady frictionless flow in a confined region, for which the velocity distribution cannot be obtained purely on the basis of an *inviscid* fluid model. Batchelor (1956) has discussed this latter problem and has shown how this indeterminancy can be overcome by the use of an integral condition arising from the effect of viscosity, no matter how small it may be. In the steady two-dimensional flow of an inviscid fluid it is well-known that the vorticity is constant along streamlines, but that it may vary from one streamline to another. Viscous effects allow the vorticity to diffuse across the streamlines until a uniform value is reached. A similar situation appears to exist in the case of magnetohydrodynamic flow, except that one must consider the effect of finite electrical conductivity. Unfortunately the equations do not yield any simple integral condition of the type found by Batchelor. A fuller analysis is required which takes into account the perturbations due to large but finite conductivity. It is hoped to pursue this matter further, and if necessary to solve numerically for the unsteady flow of a fluid of finite electrical conductivity.

One final comment should be made. Equation (8) shows that at the surface of the obstacle the magnetic field $\mathbf{H}$ will be zero; the current density $\mathbf{j}$ will in general be non-zero. However a glance at Ohm's law, equation (3), with $\mathbf{E} \equiv 0$, shows that, if large but finite electrical conductivity had been considered, $\mathbf{j} = 0$ on the surface. Thus a magnetic boundary layer must be considered near the surface in which the magnetic field diffuses in such a way that $\mathbf{j}$ becomes zero at the surface. This has been discussed by Glauert (1961), who was able to show that for the flow past a flat plate the perfect conductivity solution gives the correct limiting skin friction. The tangential component of the magnetic field at the surface was found to be of $O(\epsilon^{-\frac{1}{2}})$ as $\epsilon = \mu\sigma\nu$ becomes large.

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